(b) For an arithmetic series,

$$S_n = \frac{n}{2}(t_1 + t_n).$$

Thus, $$S_9 = \tfrac{9}{2}(t_1 + t_9) = \tfrac{9}{2}(8 + (-4)) = 18.$$

Example 3. A geometric series has $t_2 = 6$, $t_5 = 48$. Find (a) t_n, (b) S_{10}.

Solution: (a) Let the common ratio be r, so that $t_n = t_1 \cdot r^{n-1}$.

Then we have
$$6 = t_1 \cdot r$$
$$48 = t_1 \cdot r^4,$$
giving
$$r^3 = \tfrac{48}{6} = 8.$$
Thus,
$$r = 2$$
$$t_1 = 3$$
and
$$t_n = 3 \cdot 2^{n-1}.$$

(b) For a geometric series, $S_n = \dfrac{t_1(r^n - 1)}{r - 1}$.

When $t_1 = 3$, $r = 2$, and $n = 10$, we have:

$$S_{10} = \frac{3(2^{10} - 1)}{2 - 1}$$
$$= 3(1024 - 1)$$
$$= 3069.$$

Exercises [A-1]

1. A sequence is defined by $t_n = \dfrac{3n}{2n+1}$.

   (a) Is $\frac{33}{23}$ a term of the sequence? (b) Is $\frac{21}{13}$ a term of the sequence?
2. An arithmetic sequence has $t_1 = 6$, $t_2 = 10$. Find (a) t_n, (b) t_6.
3. Find an expression for t_n in terms of n for the arithmetic sequence 4, 7, 10, $\cdots$.
4. If $t_n = 6n + 1$, show that $t_{i+1} = t_i + 6$.
5. Find the number of multiples of 7 between 22 and 431.
6. Find the first term and common difference of the arithmetic sequence in which $t_5 = -3$, $t_9 = -11$.
7. An arithmetic sequence has $t_{18} = 3t_5$. Find the common difference d in terms of t_1.
8. A geometric sequence has $t_3 = 10$, $t_4 = 2$. Find (a) t_n, (b) t_1.
9. Find an expression for t_n in terms of n for the geometric sequence 4, 8, 16, $\cdots$.
10. If a geometric sequence has first term 2 and common ratio 3, find (a) t_n, (b) t_6.

11. An arithmetic series has $t_1 = 5$, $t_2 = 8$. Find (a) t_n, (b) S_n, (c) t_{11}, (d) S_{10}.

12. Find the sum of all positive integers less than 150 which are divisible by 4.

13. If $t_n = 4n + 1$ for every positive integer n, find S_n.

14. If $S_n = n^2 + 4n$ for every positive integer n, find t_n. (*Hint:* $S_1 = t_1$ and $S_n - S_{n-1} = t_n$.)

15. An infinite series has nth partial sum $S_n = \dfrac{n^2}{n^2+1}\cdot$ Find the first five terms of the series. Find an expression for t_n in terms of n. (Note that $t_1 = S_1$, $t_2 = S_2 - S_1$, etc.)

16. Show that if $t_n = 2^{n-1}$, then $S_n = 2^n - 1$.

17. Show that if $t_n = 2^{1-n}$, then $S_n = 2 - 2^{1-n}$. Find an approximation to the value of $| S_n - 2 |$ when $n = 21$. ($2^{10} \approx 1000$.)

18. A geometric series has first term 2 and common ratio $\frac{2}{3}$. Show that $S_n < 6$ for all n.

19. If $\{t_n\}$ represents a geometric sequence with $t_n = a \cdot r^{n-1}$, $a > 0$, $r > 0$, show that $\{\log t_n\}$ represents an arithmetic sequence. NOTE: We shall often use the notation $\{t_n\}$ to represent a sequence $t_1, t_2, t_3, \cdots, t_n, \cdots$.

20. Write the values of S_1, S_2, S_3, S_4 for the infinite series

$$\ln 2 + \ln \tfrac{3}{2} + \ln \tfrac{4}{3} + \ln \tfrac{5}{4} + \cdots + \ln\left(\frac{n+1}{n}\right) + \cdots.$$

Find an expression for S_n in terms of n. Use mathematical induction to confirm your result. NOTE: The symbol $\ln x$ is a common abbreviation for $\log_e x$.

Exercises [A-2]

1. (a) Find the first five terms of the sequence defined by $t_n = \dfrac{1}{2n-1}\cdot$

   (b) Find the first eight terms of the sequence defined by

$$t_n = \begin{cases} 1, & n \text{ odd} \\ \dfrac{1}{n+1}, & n \text{ even.} \end{cases}$$

2. An arithmetic sequence has $t_3 = 4$, $t_4 = 1$. Find (a) t_n, (b) t_1.

3. Find a linear expression for t_n in terms of n for the arithmetic sequence with $t_1 = 3$, $t_2 = 6$.

4. If $t_n = 5 + 6n$, find t_3, t_4, and $t_{n+1} - t_n$.

5. In an arithmetic sequence, $t_9 = 21$, $t_{18} = 48$. Find t_{27}.

6. If $t_{12} = 2t_6$ in an arithmetic sequence, find t_1 in terms of the common difference d.

7. Show that in any arithmetic progression, $t_n + t_1 = t_{n-1} + t_2 = t_{n-2} + t_3$.

8. Find t_6 for the geometric sequence $3, -\frac{3}{2}, \frac{3}{4}, \cdots$.

9. A geometric sequence has $t_1 = 3$, $t_2 = 6$. Find (a) t_n, (b) t_6.

10. If $t_2 = 6$ and $t_5 = \frac{3}{4}$ in a geometric sequence, find an expression for t_n in terms of n.

11. An arithmetic series has $t_1 = -8$, $t_2 = -2$. Find (a) t_n, (b) S_n, (c) the value of n for which $S_n = 300$.

12. Find the sum of all positive integers less than 1000 which are divisible by 3.

13. Find S_n if (a) $t_n = n$, (b) $t_n = 2n - 1$.

14. Find the first five terms of the infinite series which has nth partial sum $S_n = \dfrac{1}{2^n}$.

15. An infinite series has nth partial sum $S_n = 2^n$. Find the first five terms of the series.

16. A geometric series has first term 4, common ratio 3. Find S_1, S_2, S_3, S_4.

17. A geometric series has $t_1 = 5$, $t_2 = -10$. Find (a) t_8, (b) S_8.

18. A geometric series has first term 6, common ratio $\frac{1}{3}$. Find S_1, S_2, S_3, S_4. Is there an n such that $S_n > 9$?

19. A geometric series has $t_3 = 1$, $t_6 = \frac{1}{8}$. Show that $S_n < 8$ for all n.

20. Prove that if a, b, c are successive terms of an arithmetic sequence $(a \neq b)$, then a, b, c cannot be successive terms of a geometric sequence.

13.3 Limits of Infinite Sequences

In the preceding section we were concerned mainly with expressing t_n and S_n in terms of n and then finding values of t_n and S_n for specific positive integers n. In this section we shall focus our attention on the general nature of the nth term of an infinite sequence. We shall be particularly interested in the question of whether the nth term of a given sequence approximates a fixed number more and more closely as n increases.

As we have already noted, associated with any infinite series $t_1 + t_2 + t_3 + \cdots + t_n + \cdots$ is its sequence of partial sums, $S_1, S_2, S_3, \cdots, S_n, \cdots$ where $S_n = \sum_{i=1}^{n} t_i$. As an example, consider the geometric series $4 + 2 +$

Pages 406–408

1. (a) 30 (b) $\sum_{i=1}^{10} 2i$ 3. (a) 99 (b) $\sum_{i=10}^{100} i^2$

5. (a) $\sum_{i=1}^{n} (2i-1) = n^2$ (b) $2n+1$ 7. (a) even integer (b) odd integer

9. (a) $3 \mid p$ (b) $3 \nmid p$ 11. $(n+1)^2$ 13. $2n+1$ 15. $f(n+1)$

Pages 415–416

1. (a) 1 (b) 0 (c) none (d) none (e) 3 (f) none (g) none (h) 2 (i) 4 (j) none (k) none
3. (a) yes (b) yes 5. (a) yes (b) yes

Pages 429–430

1. (a) yes (b) no 3. $3n+1$ 5. 58 7. $\frac{2}{5} t_1$ 9. 2^{n+1}

11. (a) $3n+2$ (b) $\frac{n}{2}(3n+7)$ (c) 32 (d) 185 13. $n(2n+3)$

15. $\frac{1}{2}, \frac{3}{10}, \frac{1}{10}, \frac{7}{170}, \frac{9}{442}; \frac{2n-1}{(n^2-1)(n^2-2n+2)}$

Pages 430–431

1. (a) $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}$ (b) $1, \frac{1}{3}, 1, \frac{1}{5}, 1, \frac{1}{7}, 1, \frac{1}{9}$ 3. $3n$ 5. 75
9. (a) $3(2)^{n-1}$ (b) 96 11. (a) $6n-14$ (b) $n(3n-11)$ (c) 12

13. (a) $\frac{n}{2}(1+n)$ (b) n^2 15. 2, 2, 4, 8, 16 17. (a) 640 (b) -425

Pages 436–438

1. 0 3. 0 5. (a) (d) (e) Convergent, (b) (c) (f) Divergent

9. (a) $n > \frac{6}{\log_{10} 2}$ (b) $n > \frac{10}{\log_{10} 1.2}$ 11. (a) 1 (b) $\frac{c}{1-x}$

Pages 441–443

1. (a) 1 (b) 4 (c) 50 (d) $\frac{50}{9}$ (e) 1 3. $\frac{1}{6}$ 7. $\frac{1}{5}$ 9. no 13. $\frac{1}{2}$
21. 1 23. $\frac{1}{2}$

Pages 446–447

1. $0.\overline{09}$ 3. $0.\overline{1176470588235294}$ 5. (a) $\frac{3}{8}$ (b) $\frac{62}{165}$ 9. $\frac{5}{6}$
15. (a) $\frac{5}{8}$ (b) $\frac{21}{32}$ 17. (a) $\frac{1}{99}$ (b) $\frac{1}{8}$ (c) $\frac{1}{3}$